# Complex asymptotics of Poincaré functions and properties of Julia sets

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*Dedicated to Robert F. Tichy on the occasion of his* $50^{th}$ *birthday.*

*Abstract*

The asymptotic behaviour of the solutions of Poincaré's functional equation $f(\lambda z) = p(f(z))$ $(\lambda > 1)$ for $p$ a real polynomial of degree $\geq 2$ is studied in angular regions $W$ of the complex plain. It is known [**9**, **10**] that $f(z) \sim \exp(z^\rho F(\log_\lambda z))$, if $f(z) \to \infty$ for $z \to \infty$ and $z \in W$, where $F$ denotes a periodic function of period 1 and $\rho = \log_\lambda \deg(p)$. In the present paper we refine this result and derive a full asymptotic expansion. The constancy of the periodic function $F$ is characterised in terms of geometric properties of the Julia set of $p$. For real Julia sets we give inequalities for multipliers of Pommerenke-Levin-Yoccoz type. The distribution of zeros of $f$ is related to the harmonic measure on the Julia set of $p$.

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## 1. *Introduction*

### 1·1. *Historical remarks*

In his seminal papers [**36**, **37**] H. Poincaré has studied the equation

$$f(\lambda z) = R(f(z)), \quad z \in \mathbb{C}, \tag{1.1}$$

where $R(z)$ is a rational function and $\lambda \in \mathbb{C}$. He proved that, if $R(0) = 0$, $R'(0) = \lambda$, and $|\lambda| > 1$, then there exists a meromorphic or entire solution of (1.1). After Poincaré, (1.1) is called *the Poincaré equation* and solutions of (1.1) are called *the Poincaré functions* . The next important step was made by G. Valiron [**45**, **46**], who investigated the case, where $R(z) = p(z)$ is a polynomial, i.e.

$$f(\lambda z) = p(f(z)), \quad z \in \mathbb{C}, \tag{1.2}$$

and obtained conditions for the existence of an entire solution $f(z)$. Furthermore, he derived the following asymptotic formula for $M(r) = \max_{|z|\le r} |f(z)|$:

$$\log M(r) \sim r^\rho F\left(\frac{\log r}{\log|\lambda|}\right), \quad r \to \infty. \tag{1.3}$$

Here $F(z)$ is a 1-periodic function bounded between two positive constants, $\rho = \frac{\log d}{\log|\lambda|}$ and $d = \deg p(z)$.

Different aspects of the Poincaré functions have been studied in the papers [**9**, **10**, **12**, **14**, **21**, **41**]. In particular in [**9**], in addition to (1.3), asymptotics of entire solutions $f(z)$ on various rays $re^{i\vartheta}$ of the complex plane have been found.

It turns out that this asymptotic behaviour heavily depends on the arithmetic nature of $\lambda$. For instance, if $\arg\lambda = 2\pi\beta$, and $\beta$ is irrational, then $f(z)$ is unbounded along any ray $\arg z = \vartheta$ (cf. [**9**]).

### 1·2. *Assumptions*

In the present paper we concentrate on the simplest, but maybe most important case for applications, namely, when $\lambda$ is real and $p(z)$ is a real polynomial (i. e. all coefficients of $p(z)$ are real).

It is known from [**46**] and [**9**] that, if $f(z)$ is an entire solution of (1.2), then the only admissible values for $f_0 = f(0)$ are the fixed points of $p(z)$ (i. e. $p(f_0) = f_0$). Moreover, entire solutions exist, if and only if there exists an $n_0 \in \mathbb{N}$ such that

$$\lambda^{n_0} = p'(f_0).$$

It was proved in [**9**, Propositions 2.1–2.3] that the general case may be reduced to the simplest case

$$f(0) = p(0) = 0 \text{ and } p'(0) = \lambda > 1$$

by a change of variables. In the same vein, we can assume without loss of generality that $f'(0) = 1$ and the polynomial $p$ is monic (i. e. the leading coefficient is 1)

$$p(z) = z^d + p_{d-1}z^{d-1} + \cdots + p_1 z. \tag{1.4}$$

### 1·3. *Poincaré and Schröder equations*

The functional equation (1.2) with the additional (natural) conditions $f(0) = 0$ and $f'(0) = 1$ is closely related to Schröder's functional equation (cf. [**42**])

$$g(p(z)) = \lambda g(z), \quad g(0) = 0 \text{ and } g'(0) = 1 \tag{1.5}$$

which was used by G. Koenigs [**22**, **23**] to study the behaviour of $p$ under iteration around the repelling fixed point $z = 0$. By definition, $g$ is the local inverse of $f$ around $z = 0$. Both functions together provide a linearisation of $p$ around its repelling fixed point $z = 0$

$$g(p(f(z))) = \lambda z \text{ and } g(p^{(n)}(f(z))) = \lambda^n z,$$

where $p^{(n)}(z)$ denotes the $n$-th iterate of $p$ given by $p^{(0)}(z) = z$ and $p^{(n+1)}(z) = p(p^{(n)}(z))$.

We note here that (1.1) and (1.2) are also called Schröder equation by some authors. For instance, the value distribution of solutions of the Poincaré (alias Schröder) equation (1.1) has been investigated in [**21**].

### 1·4. *Branching processes and diffusion on fractals*

Iterative functional equations occur in the context of branching processes (cf. [**20**]). Here a probability generating function

$$q(z) = \sum_{n=0}^{\infty} p_n z^n$$

encodes the offspring distribution, where with $p_n \geq 0$ is the probability that an individual has $n$ offspring in the next generation (note that $q(1) = 1$). The growth rate $\lambda = q'(1)$ decides whether the population is increasing ($\lambda > 1$) or dying out $\lambda \leq 1$. In the first case the branching process is called *super-critical*. The probability generating function $q^{(n)}(z)$ ($n$-th iterate of $q$) encodes the distribution of the size $X_n$ of the $n$-th generation under the offspring distribution $q$. In the case of a super-critical branching process it is known that the random variables $\lambda^{-n}X_n$ tend to a limiting random variable $X_\infty$. The moment generating function of this random variable

$$f(z) = \mathbb{E}e^{-zX_\infty}$$

satisfies the functional equation (cf. [**20**])

$$f(\lambda z) = q(f(z)),$$

which is (1.2), if $q$ is a polynomial. Furthermore, this equation can be transformed into (1.2), if $q$ is conjugate to a polynomial by a Möbius transformation, especially $q(z) = \frac{1}{p(1/z)}$, where $p$ is a polynomial.

Branching processes have been used in [**1**, **2**, **30**] to model time for the Brownian motion on certain types of self-similar structures such as the Sierpiński gasket. In this context the zeros of the solution of (1.2) are the eigenvalues of the infinitesimal generator of the diffusion ("Laplacian"), if the generating function of the offspring distribution is conjugate to a polynomial (cf. [**10**, **18**, **31**, **43**, **44**]). In this case the zeros of $f$ have to be real, since they are eigenvalues of a self-adjoint operator. This motivates the investigation of real Julia sets in Section 4.

### 1·5. *Contents*

The paper is organised as follows.

In Section 2·1 we study the asymptotic behaviour of $f(z)$ in those sectors $W$ of the complex plane, where

$$f(z) \to \infty \text{ for } z \to \infty, \quad z \in W. \tag{1.6}$$

It was proved in [**9**, **10**] that (1.6) implies

$$f(z) \sim \exp\left(z^\rho F\left(\frac{\log z}{\log\lambda}\right)\right) \text{ for } z\to\infty, \quad z\in W,$$

where $F(z)$ is a periodic function of period 1. In Section 2·1 we will refine this result to a full asymptotic expansion of $f(z)$, which takes the form

$$f(z) = \exp\left(z^\rho F\left(\log_\lambda z\right)\right) + \sum_{n=0}^{\infty} c_n \exp\left(-nz^\rho F\left(\log_\lambda z\right)\right), \tag{1.7}$$

where $F$ is a periodic function of period 1 holomorphic in some strip depending on $W$ and $\rho = \log_\lambda d$. The proof is based on an application of the Böttcher function at $\infty$ of $p(z)$.

We note here that E. Romanenko and A. Sharkovsky [**41**] have studied equation (1.2) on $\mathbb{R}$ (rather than $\mathbb{C}$) and obtained a full asymptotic expansion of this type by Sharkovsky's method of "first integrals" or "invariant curves".

Further analysis of the periodic function $F$ occurring in (1.7) is presented in Section 2·3, where the Fourier coefficients of $F$ are related to the Böttcher function at $\infty$ of $p(z)$ and the harmonic measure on the Julia set of $p$.

In Section 2·4 the asymptotic behaviour of $f(z)$ is studied in sectors that are related to basins of attraction of finite attracting fixed points.

In Section 3 we relate geometric properties of the Julia set to the location of the zeros of $f$.

Section 4 is devoted to the special case of real Julia sets $\mathcal{J}(p)$. Here we prove, in particular, the following inequalities of Pommerenke-Levin-Yoccoz type for multipliers of fixed points $\xi$:

$$p(\xi) = \xi \Rightarrow \begin{cases} |p'(\xi)| \geq d & \text{for } \min\mathcal{J}(p) < \xi < \max\mathcal{J}(p) \\ |p'(\xi)| \geq d^2 & \text{for } \xi = \min\mathcal{J}(p) \text{ or } \xi = \max\mathcal{J}(p). \end{cases} \tag{1.8}$$

Furthermore, equality can hold only, if $p$ is linearly conjugate to a Chebyshev polynomial of the first kind.

In Section 5 we continue the study of Dirichlet generating functions of zeros of Poincaré functions that we started in [**10**] in the context of spectral zeta functions on certain fractals. We relate the poles and residues of the zeta function of $f$ to the Mellin transform of the harmonic measure $\mu$ on the Julia set of $p$. Furthermore, we show a connection between the zero counting function of $f$ and the harmonic measure $\mu$ of circles around the origin.

## 2. *Relation of complex asymptotics and the Fatou set*

Throughout the rest of the paper we will use the following notations and assumptions. Let $p$ be a real polynomial of degree $d$ as in (1.4). We always assume that $p(0) = 0$ and $p'(0) = a_1 = \lambda$ with $|\lambda| > 1$. We refer to [**3**, **33**] as general references for complex dynamics.

We denote the Riemann sphere by $\mathbb{C}_\infty$ and consider $p$ as a map on $\mathbb{C}_\infty$. We recall that the Fatou set $\mathcal{F}(p)$ is the set of all $z \in \mathbb{C}_\infty$ which have an open neighbourhood $U$ such that the sequence $(p^{(n)})_{n\in\mathbb{N}}$ is equicontinuous on $U$ in the chordal metric on $\mathbb{C}_\infty$. By

definition $\mathcal{F}(p)$ is open. We will especially need the component of $\infty$ of $\mathcal{F}(p)$ given by

$$\mathcal{F}_\infty(p) = \left\{z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = \infty\right\}, \tag{2.1}$$

as well as the basins of attraction of a finite attracting fixed point $w_0$ ($p(w_0) = w_0$, $|p'(w_0)| < 1$)

$$\mathcal{F}_{w_0}(p) = \left\{z \in \mathbb{C} \mid \lim_{n\to\infty} p^{(n)}(z) = w_0\right\}. \tag{2.2}$$

The complement of the Fatou set is the Julia set $\mathcal{J}(p) = \mathbb{C}_\infty \setminus \mathcal{F}(p)$.

The filled Julia set is given by

$$\mathcal{K}(p) = \left\{z \in \mathbb{C} \mid (p^{(n)}(z))_{n\in\mathbb{N}} \text{ is bounded}\right\} = \mathbb{C} \setminus \mathcal{F}_\infty(p). \tag{2.3}$$

Furthermore, it is known that (cf. [**15**])

$$\partial\mathcal{K}(p) = \partial\mathcal{F}_\infty(p) = \mathcal{J}(p). \tag{2.4}$$

In the case of polynomials this can be used as an equivalent definition of the Julia set.

We will also use the notations

$$W_{\alpha,\beta} = \{z \in \mathbb{C} \setminus \{0\} \mid \alpha < \arg z < \beta\} \tag{2.5}$$

and

$$B(z, r) = \{w \in \mathbb{C} \mid |z - w| < r\}.$$

2·1. *Asymptotics in the infinite Fatou component*

In [**9**, **10**] the asymptotics of the solution of the Poincaré equation (1.2) was given. We want to present a different approach here, which gives a full asymptotic expansion.

Theorem 2·1. *Let $f$ be the entire solution of the Poincaré equation* (1.2) *for a real polynomial $p$ with $\lambda = p'(0) > 1$. Assume further that the Fatou component of $\infty$, $\mathcal{F}_\infty(p)$ contains an angular region $W_{\alpha,\beta}$.*

**A** *Then the following asymptotic expansion for $f$ is valid for all $z \in W_{\alpha,\beta}$ large enough*

$$f(z) = \exp\left(z^\rho F\left(\log_\lambda z\right)\right) + \sum_{n=0}^{\infty} c_n \exp\left(-nz^\rho F\left(\log_\lambda z\right)\right), \tag{2.6}$$

*where $F$ is a periodic function of period 1 holomorphic in the strip*

$$\left\{z \in \mathbb{C} \mid \frac{\alpha}{\log\lambda} < \Im z < \frac{\beta}{\log\lambda}\right\}$$

*and $\rho = \log_\lambda d$. Furthermore,*

$$\forall z \in W_{\alpha,\beta} : \Re z^\rho F(\log_\lambda z) > 0 \tag{2.7}$$

*holds.*

**B** *Let $g$ denote the Böttcher function associated with $p$,* i. e.

$$(g(z))^d = g(p(z)) \tag{2.8}$$

*in some neighbourhood of $\infty$. Its inverse function is given by the Laurent series around $\infty$*

$$g^{(-1)}(w) = w + \sum_{n=0}^{\infty} \frac{c_n}{w^n}. \tag{2.9}$$

*Then we have*

$$f(z) = g^{(-1)}\left(\exp\left(z^\rho F\left(\log_\lambda z\right)\right)\right)$$

*and $c_n$ can be determined from the coefficients of $p$.*

*Proof.* We recall that $p$ has a super-attracting fixed point of order $d = \deg p$ at infinity. We consider the Böttcher function $g$ associated with this fixed point (cf. [**3**, **5**, **6**, **26**]), which satisfies the functional equation (2.8) in some neighbourhood of infinity. The Böttcher function has a Laurent expansion around infinity given by

$$g(z) = z + \sum_{n=0}^{\infty} \frac{b_n}{z^n}, \tag{2.10}$$

which converges for $|z| > R$ for some $R > 0$. The coefficients $(b_n)_{n\in\mathbb{N}_0}$ can be determined uniquely from the coefficients of the polynomial $p$.

Using the Böttcher function we can rewrite the Poincaré equation assuming that $|f(z)| > R$

$$(g(f(z)))^d = g(p(f(z))) = g(f(\lambda z)). \tag{2.11}$$

From this we derive that $h(z) = g(f(z))$ satisfies the much simpler functional equation

$$(h(z))^d = h(\lambda z),$$

which only holds for those values $z$ for which $|f(z)| > R$. This equation has solutions

$$h(z) = \exp\left(z^\rho F\left(\log_\lambda z\right)\right) \tag{2.12}$$

with $\rho = \log_\lambda d$ and $F$ a periodic function of period 1 holomorphic in some strip parallel to the real axis. Since $|h(z)| > 1$ for all $z$ with $|f(z)| > R$ by the properties of the function $g$, we have (2.7).

By (2.10) $g$ is invertible in some neighbourhood of $\infty$ and we can write (2.9) where the coefficients $c_n$ depend only on the coefficients of the polynomial $p$. This function satisfies the functional equation

$$g^{(-1)}(w^d) = p(g^{(-1)}(w)) \tag{2.13}$$

for $w$ in some neighbourhood of $\infty$. Inserting (2.12) into (2.9) yields (2.6) giving an exact and asymptotic expression for $f(z)$. $\square$

*Remark* 2·1. E. Romanenko and A. Sharkovsky have studied equation (1.2) on $\mathbb{R}$ (rather than on $\mathbb{C}$) in [**41**]. Applying Sharkovsky's method of "first integrals" ("invariant graphs") they obtained a full asymptotic formula of type (2.6) for all solutions $f(x)$, such that $f(x) \to \infty$ for $x \to \infty$.

### 2·2. *Böttcher functions, Green functions, and constancy of the periodic function $F$*

We will make frequent use of the integral representation of the Böttcher function

$$g(z) = \exp\left(\int_{\mathcal{J}(p)} \log(z-x)\, d\mu(x)\right), \tag{2.14}$$

where $\mu$ denotes the harmonic measure on the Julia set $\mathcal{J}(p)$ (cf. [**4**, **7**, **39**]). This shows that $g$ is holomorphic on any simply connected subset of $\mathcal{F}_\infty(p)$. The measure $\mu$ can be

given as the weak limit of the measures

$$\mu_n = \frac{1}{d^n} \sum_{p^{(n)}(x)=\xi} \delta_x, \tag{2.15}$$

where $\xi$ can be chosen arbitrarily (not exceptional) and $\delta_x$ denotes the unit point mass at $x$ (cf. [**7**, **39**]).

The function $g(z)$ can be continued to any simply connected subset $U$ of $\mathbb{C}_\infty \setminus \mathcal{K}(p)$ (this follows for instance from the integral representation (2.14)). Furthermore, it follows from [**3**, Lemma 9.5.5] and (2.8) that

$$g(U) \subset \{z \in \mathbb{C}_\infty \mid |z| > 1\}.$$

The function $\log|g(z)|$ is the Green function for the logarithmic potential on $\mathcal{F}_\infty(p)$ (cf. [**3**, Section 9]). Combining classical potential theory with polynomial iteration theory we get

$$\lim_{\substack{z \to z_0 \\ z \in \mathcal{F}_\infty(p)}} |g(z)| = 1 \Leftrightarrow z_0 \in \mathcal{J}(p), \tag{2.16}$$

where the implication $\Leftarrow$ is [**3**, Lemma 9.5.5]. The opposite implication is a general property of the Green function (cf. [**16**, Chapter III], and [**39**, Section 6.5]) combined with the fact that $\partial\mathcal{F}_\infty(p) = \mathcal{J}(p)$ for polynomial $p$.

THEOREM 2·2. *The periodic function $F$ occurring in the asymptotic expression* (2.6) *for $f$ is constant, if and only if the polynomial $p$ is either linearly conjugate to $z^d$ or to the Chebyshev polynomial of the first kind $T_d(z)$.*

*Proof.* The periodic function $F$ is constant, if and only if the function $h(z) = g(f(z))$ introduced above satisfies

$$h(z) = \exp(Cz^\rho) \tag{2.17}$$

for some constant $C \neq 0$. This implies that for any $w_0 \in \mathcal{J}(p) \setminus \{0\}$ the function $g$ has an analytic continuation to some open neighbourhood of $w_0$. Thus (2.16) can be replaced by

$$|g(w_0)| = 1 \Leftrightarrow w_0 \in \mathcal{J}(p)$$

in our case. By (2.17) this is equivalent to $w_0 = f(z_0)$ for $Cz_0^\rho \in i\mathbb{R}$. Since $Cz^\rho \in i\mathbb{R}$ describes an analytic curve (with a possible cusp at $z = 0$), the Julia set of $p$ is the image of this curve under the entire function $f$, thus itself an analytic arc.

By [**19**, Theorem 1] $\mathcal{J}(p)$ can only be an analytic arc, if the Julia set of $p$ is either a line segment or a circle. The Julia set is a line segment, if and only if $p$ is linearly conjugate to the Chebyshev polynomial $T_d$ (cf. [**3**, Theorem 1.4.1]); the Julia set is a circle, if and only if $p$ is linearly conjugate to $z^d$ (cf. [**3**, Theorem 1.3.1]). $\square$

*Remark* 2·2. Suppose that the periodic function $F$ is constant. If $p$ is linearly conjugate to a monomial, then the Böttcher function $g$ and therefore its inverse are linear functions. In this case $\rho = 1$. (We recall that we generally assume that $f'(0) = 1$.) If $p$ is linearly conjugate to a Chebyshev polynomial, $g^{(-1)}$ is linearly conjugate to the Joukowski function $z + \frac{1}{z}$. In this case $\rho = 1$, if 0 is an inner point of the line segment $\mathcal{J}(p)$, and $\rho = \frac{1}{2}$, if 0 is an end point of the line segment $\mathcal{J}(p)$ (cf. Sections 4·1 and 4·2). Furthermore, the asymptotic series (2.6) is finite, if the periodic function $F$ is constant.

2·3. *Further analysis of the periodic function*

In this section we relate the periodic function $F$ occurring in (2.6) to the local behaviour of the Böttcher function at the fixed point $f(0) = 0$.

This will allow to express the Fourier coefficients of $F$ in terms of residues of the Mellin transform (cf. [**11**, **34**]) of the harmonic measure $\mu$ given by (2.15). This Mellin transform was introduced and studied in [**4**]. A similar relation was also used in [**18**] to derive an asymptotic expression for $f$ in a special case.

We will use the relation

$$G(w) = \log g(w) = \int_{\mathcal{J}(p)} \log(w - x)\, d\mu(x) \tag{2.18}$$

between the (complex) "Green function" $G$ and the Böttcher function $g$. Assume that the Fatou component $\mathcal{F}_\infty(p)$ contains an angular region centred at the fixed point 0. Furthermore, assume that $\lim_{w\to 0} g(w) = 1$. Then (2.12) holds in this angular region. This fact can be used to analyse the local behaviour of $\log g(w)$ around $w = 0$:

$$\log g(w) = \left(f^{(-1)}(w)\right)^\rho F\left(\log_\lambda f^{(-1)}(w)\right) = w^\rho F\left(\log_\lambda w\right) + \mathcal{O}(w^{\rho+1}). \tag{2.19}$$

Thus the behaviour of the Green function $G$ at the point 0 exhibits the same periodic function $F$ as the asymptotic expansion of $\log f$ around $\infty$.

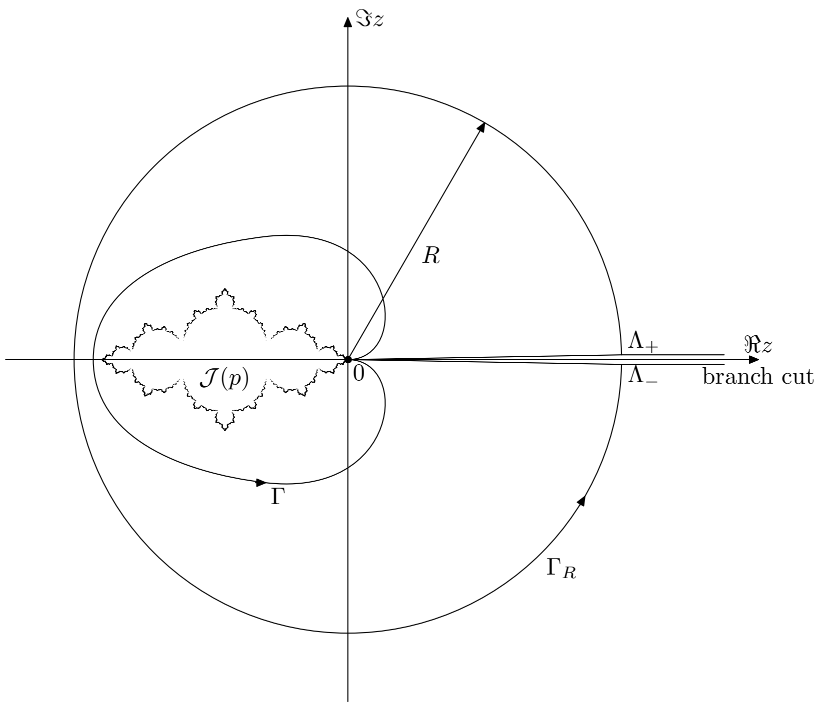


Fig. 1. Paths of integration.

We now relate the Green function $G(w)$ to the Mellin transform of $\mu$

$$M_\mu(s) = \int_{\mathcal{J}(p)} (-x)^s\, d\mu(x), \tag{2.20}$$

where the branch cut for the function $(-x)^s$ is chosen to connect 0 with $\infty$ without any

further intersection with $\mathcal{J}(p)$. Following the computations in [**4**, Section 5] we obtain

$$M_\mu(s) = \frac{1}{2\pi i}\oint_\Gamma (-z)^s\,dG(z) = \frac{1}{2\pi i}\oint_{\Gamma_R} (-z)^s\,dG(z).$$

For $\Re s < 0$ we have for the circle of radius $R$

$$\left|\frac{1}{2\pi i}\int_{|z|=R} (-z)^s\,dG(z)\right| \ll R^{\Re s},$$

which allows to let $R \to \infty$ in this case. This gives

$$M_\mu(s) = \frac{1}{2\pi i}\left(\int_{\Lambda_+} (-z)^s\,dG(z) - \int_{\Lambda_-} (-z)^s\,dG(z)\right)$$
$$= \frac{e^{-i\pi s} - e^{i\pi s}}{2\pi i}\int_0^\infty x^s G'(x)\,dx = s\frac{\sin \pi s}{\pi}\int_0^\infty x^{s-1}G(x)\,dx,$$

which relates the Mellin transform of the measure $\mu$ to the Mellin transform of the function $G(z)$

$$\mathcal{M}G(s) = \int_0^\infty x^{s-1}G(x)\,dx = \frac{\pi}{s\sin\pi s}M_\mu(s) \text{ for } -\rho < \Re s < 0. \tag{2.21}$$

The function $M_\mu(s)$ (and therefore $\mathcal{M}G(s)$ by (2.21)) has an analytic continuation by the following observation

$$M_\mu(s) = \frac{1}{d}\sum_{k=1}^d \int_{\mathcal{J}(p)} (-p_k^{(-1)}(x))^s\,d\mu(x), \tag{2.22}$$

where $p_k^{(-1)}$ $(k = 1, \ldots, d)$ denote the $d$ branches of the inverse function of $p$; we choose the numbering so that $p_1^{(-1)}(0) = 0$. The summands for $k = 2, \ldots, d$ are clearly entire functions in $s$, since the integrand is bounded away from 0 and $\infty$. For the summand with $k = 1$ we observe that

$$p_1^{(-1)}(x) = \frac{1}{\lambda}x + \mathcal{O}(x^2) \text{ for } x \to 0. \tag{2.23}$$

Inserting this into (2.22) gives

$$M_\mu(s) = \frac{1}{d}\lambda^{-s}\int_{\mathcal{J}(p)} (-x)^s\,d\mu(x) + \frac{1}{d}\lambda^{-s}\int_{\mathcal{J}(p)} (-x)^s\mathcal{O}(x)\,d\mu(x)$$
$$+ \frac{1}{d}\sum_{k=2}^d \int_{\mathcal{J}(p)} (-p_k^{(-1)}(x))^s\,d\mu(x),$$

where the second term on the right-hand-side originates from inserting the holomorphic function $\mathcal{O}(x^2)$ from (2.23) into the integrand, which gives a function holomorphic in a larger domain. Thus we obtain

$$M_\mu(s) = \frac{1}{d\lambda^s - 1}H(s) \tag{2.24}$$

for some function $H(s)$ holomorphic for $\Re s > -\rho - 1$ $(\rho = \log_\lambda d)$. The numerator $d\lambda^s - 1$ has zeros at $s = -\rho + \frac{2k\pi i}{\log\lambda}$ $(k \in \mathbb{Z})$, which give possible poles for the function $M_\mu(s)$.

*Remark* 2·3. Using the full Taylor expansion of $p_1^{(-1)}(x)$ instead of the $\mathcal{O}$-term in (2.23)

would yield the existence of a meromorphic continuation of $M_\mu(s)$ to the whole complex plane.

Taking (2.21) and (2.24) together gives the analytic continuation of $\mathcal{M}G(s)$ to $-\rho-1<\Re s<0$. Then the Mellin inversion formula (cf. [**11**]) gives (for $-\rho<c<0$)

$$\begin{aligned} G(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\mathcal{M}G(s)x^{-s}\,ds = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\pi}{s\sin\pi s}\frac{1}{d\lambda^s-1}H(s)x^{-s}\,ds \\ &= \frac{1}{2\pi i}\int_{-\rho-\frac12-i\infty}^{-\rho-\frac12+i\infty}\frac{\pi}{s\sin\pi s}\frac{1}{d\lambda^s-1}H(s)x^{-s}\,ds + \sum_{k\in\mathbb{Z}}\operatorname*{Res}_{s=-\rho+\frac{2k\pi i}{\log\lambda}}\mathcal{M}G(s)x^{-s}. \end{aligned} \tag{2.25}$$

The integral in the second line is $\mathcal{O}(x^{\rho+\frac12})$, the sum of residues can be evaluated further to give the Fourier expansion of the periodic function $F$

$$\sum_{k\in\mathbb{Z}}\operatorname*{Res}_{s=-\rho+\frac{2k\pi i}{\log\lambda}}\mathcal{M}G(s)x^{-s} = x^\rho\sum_{k\in\mathbb{Z}}f_k e^{2k\pi i\log_\lambda x} = x^\rho F(\log_\lambda x). \tag{2.26}$$

The Fourier coefficients $f_k$ are given by

$$\begin{aligned} f_k = \operatorname*{Res}_{s=-\rho-\frac{2k\pi i}{\log\lambda}}\mathcal{M}G(s) &= \frac{\pi}{\left(-\rho-\frac{2k\pi i}{\log\lambda}\right)\sin\pi\left(-\rho-\frac{2k\pi i}{\log\lambda}\right)}\operatorname*{Res}_{s=-\rho-\frac{2k\pi i}{\log\lambda}}M_\mu(s) \\ &= \frac{\pi}{(-\log d-2k\pi i)\sin\pi\left(-\rho-\frac{2k\pi i}{\log\lambda}\right)}H\left(-\rho-\frac{2k\pi i}{\log\lambda}\right). \end{aligned} \tag{2.27}$$

2·4. *Asymptotics in a finite Fatou component – analysis of asymptotic values*

It is clear from the functional equation (1.2) for $f$ that any asymptotic value of $f$ has to be an attracting fixed point of the polynomial $p$ (including $\infty$). Thus the analysis in Section 2·1 can be interpreted as the behaviour of $f$ when approaching the asymptotic value $\infty$. In the present section we extend this analysis to all asymptotic values.

First we study the case of a finite attracting, but not super-attracting fixed point. Let $w_0$ be an attracting fixed point of $p$ and denote $\eta=p'(w_0)\neq 0$ ($|\eta|<1$). Then there exists a solution $\Psi$ of the Schröder equation

$$\eta\Psi(z)=\Psi(p(z)),\quad \Psi(w_0)=0,\ \text{and}\ \Psi'(w_0)=1, \tag{2.28}$$

which is holomorphic in $\mathcal{F}_{w_0}(p)$ (for instance, the sequence $(\eta^{-n}(p^{(n)}(z)-w_0))_{n\in\mathbb{N}}$ converges to $\Psi$ on any compact subset of $\mathcal{F}_{w_0}(p)$). Assume now that $\mathcal{F}_{w_0}(p)$ contains an angular region $W_{\alpha,\beta}\cap B(0,r)$ for some $r>0$. Then by conformity of $f$ some angular region at the origin is mapped into $W_{\alpha,\beta}\cap B(0,r)$. We consider the function

$$j(z)=\Psi(f(z)),$$

which satisfies the functional equation

$$j(\lambda z)=\Psi(f(\lambda z))=\Psi(p(f(z)))=\eta\Psi(f(z))=\eta j(z). \tag{2.29}$$

This equation has the solution

$$j(z)=z^{\log_\lambda\eta}H(\log_\lambda z) \tag{2.30}$$

with some periodic function of period 1, holomorphic in some strip. This periodic function

can never be constant, since otherwise $j(z)$ would have an analytic continuation to the slit complex plane. From this it would follow that $f$ is bounded in the slit complex plane, a contradiction.

The function $\Psi$ has a holomorphic inverse around 0

$$\Psi^{(-1)}(z) = w_0 + z + \sum_{n=2}^{\infty} \psi_n z^n$$

which allows us to write

$$f(z) = \Psi^{(-1)}\left(z^{\log_\lambda \eta} H(\log_\lambda z)\right) = w_0 + z^{\log_\lambda \eta} H(\log_\lambda z) + \sum_{n=2}^{\infty} \psi_n z^{n \log_\lambda \eta} (H(\log_\lambda z))^n, \tag{2.31}$$

which is valid in the angular region $W_{\alpha,\beta}$ for $z$ large enough. This gives an exact and asymptotic expression for $f$ in an angular region.

In the case of a super-attracting fixed point $w_0$ we have $p'(w_0) = 0$. Assume that the first $k-1$ derivatives of $p$ vanish in $w_0$, but the $k$-th derivative is non-zero. Then $p(z) = (z-w_0)^k P(z)$ with $P(w_0) = A \neq 0$. We use the solution $g$ of the corresponding Böttcher equation

$$g(p(z)) = A(g(z))^k \quad g(w_0) = 0, \quad g'(w_0) = 1 \tag{2.32}$$

to linearise (1.2)

$$g(f(\lambda z)) = g(p(f(z))) = A(g(f(z)))^k.$$

Thus the function $h(z) = g(f(z))$ satisfies

$$h(\lambda z) = A(h(z))^k.$$

This equation has solutions

$$h(z) = A^{-\frac{1}{k-1}} \exp\left(z^{\log_\lambda k} L\left(\log_\lambda z\right)\right)$$

for a periodic function $L$ of period 1 and a suitable choice of the $(k-1)$-th root. Furthermore, by the fact that $\lim_{z\to\infty} h(z) = 0$ we have

$$\Re\left(z^{\log_\lambda k} L\left(\log_\lambda z\right)\right) < 0 \text{ for } f(z) \in \mathcal{F}_{w_0}(p).$$

using the local inverse of $g$ around 0 we get

$$\begin{aligned} f(z) &= g^{(-1)}\left(A^{-\frac{1}{k-1}} \exp\left(z^{\log_\lambda k} L\left(\log_\lambda z\right)\right)\right) \\ &= w_0 + A^{-\frac{1}{k-1}} \exp\left(z^{\log_\lambda k} L\left(\log_\lambda z\right)\right)(1+o(1)). \end{aligned} \tag{2.33}$$

Summing up, we have proved

THEOREM 2·3. *Let $w_0$ be an attracting fixed point of $p$ such that the Fatou component $\mathcal{F}_{w_0}(p)$ contains an angular region $W_{\alpha,\beta} \cap B(0,r)$ for some $r > 0$. Then the asymptotic behaviour of $f$ for $z \to \infty$ and $z \in W_{\alpha,\beta}$ is given by* (2.31)*, if $\eta = p'(w_0) \neq 0$, and by* (2.33)*, if $p(z) - w_0$ has a zero of order $k$ in $w_0$.*

*Remark* 2·4. The periodic function $H$ in (2.31) cannot be constant, because otherwise $f(z)$ would be bounded. The periodic function $L$ in (2.33) can only be constant, if $p$ is linearly conjugate to $z^k$, by the same arguments as in the proof of Theorem 2·2 (the case

of Chebyshev polynomials does not occur, because they only have repelling finite fixed points).

As a consequence of Ahlfors' theorem on asymptotic values (cf. [**17**]) and Valiron's theorem on the growth of $f$ (cf. [**45**, **46**]) we get an upper bound for the number of attracting fixed points of a polynomial.

Theorem 2·4. *Let $p$ be a real polynomial of degree $d > 1$ and let*

$$\gamma = \max\{|p'(z)| \mid p(z) = z\}.$$

*Then the number of (finite) attracting fixed points of $p$ is bounded by $2\log_\gamma d$, i.e.*

$$\#\{z \in \mathbb{C} \mid p(z) = z \wedge |p'(z)| < 1\} \leq 2\log_\gamma d. \tag{2.34}$$

## 3. *Zeros of the Poincaré function and Julia sets*

In this section we relate the distribution of zeros of the Poincaré function in angular regions to geometric properties of the Julia set $\mathcal{J}(p)$ of the polynomial $p$.

Theorem 3·1. *Let $p$ be a real polynomial with $p(0) = 0$ and $p'(0) = \lambda > 1$. Then the following are equivalent*

(i) $\forall r > 0 : W_{\alpha,\beta} \cap \mathcal{J}(p) \cap B(0, r) \neq \emptyset$
(ii) $W_{\alpha,\beta}$ *contains a zero of $f$.*
(iii) $W_{\alpha,\beta}$ *contains infinitely many zeros of $f$.*

*Proof.* We first remark that (ii) and (iii) are trivially equivalent, since $f(z_0) = 0$ implies that $f(\lambda^n z_0) = 0$.

For the proof of "(i)⇒ (ii)" we take $0 < \varepsilon < \frac{\beta-\alpha}{2}$ so small that

$$\forall r > 0 : W_{\alpha+\varepsilon,\beta-\varepsilon} \cap \mathcal{J}(p) \cap B(0, r) \neq \emptyset.$$

Then we take $r > 0$ so small that

$$W_{\alpha+\varepsilon,\beta-\varepsilon} \cap B(0, r) \subset f(W_{\alpha,\beta}), \tag{3.1}$$

which is possible by conformity of $f$ and $f'(0) = 1$. Since the preimages of 0 are dense in $\mathcal{J}(p)$, there exists $\eta \in W_{\alpha+\varepsilon,\beta-\varepsilon} \cap B(0, r)$ and $n \in \mathbb{N}$ such that $p^{(n)}(\eta) = 0$. By (3.1) there exists $\xi \in W_{\alpha,\beta}$ such that $f(\xi) = \eta$, from which we obtain

$$f(\lambda^n \xi) = p^{(n)}(f(\xi)) = p^{(n)}(\eta) = 0.$$

For the proof of "(iii)⇒ (i)" we take $z_0 \in W_{\alpha,\beta}$ with $f(z_0) = 0$. Then

$$\forall n \in \mathbb{N} : f(\lambda^{-n} z_0) \in \mathcal{J}(p).$$

For any $r > 0$ and $n$ large enough $f(\lambda^{-n} z_0) \in W_{\alpha,\beta} \cap B(0, r)$, which gives (i). □

Similar arguments show

Theorem 3·2. *Let $p$ be a real polynomial with $p(0) = 0$ and $p'(0) = \lambda > 1$. Then*

$$\mathcal{J}(p) \subset \mathbb{R}^- \cup \{0\} \Leftrightarrow \text{ all zeros of } f \text{ are non-positive real} \tag{3.2}$$

*and*

$$\mathcal{J}(p) \subset \mathbb{R} \Leftrightarrow \text{ all zeros of } f \text{ are real.} \tag{3.3}$$

## 4. *Real Julia set*

LEMMA 4·1. *Let $p$ be a real polynomial of degree $d > 1$. Then the Julia-set $\mathcal{J}(p)$ is real, if and only if there exists an interval $[a, b]$ such that*

$$p^{(-1)}([a,b]) \subseteq [a,b]. \tag{4.1}$$

*Proof.* Assume first that $\mathcal{J}(p) \subset \mathbb{R}$ and take the interval $[a, b] = [\min \mathcal{J}(p), \max \mathcal{J}(p)]$. Let $\varepsilon > 0$. Since $\mathcal{J}(p)$ is perfect, there exist $\xi, \eta \in \mathcal{J}(p)$ with $a < \xi < a+\varepsilon < b-\varepsilon < \eta < b$. All preimages of $\xi$ and $\eta$ are in $\mathcal{J}(p)$ by the invariance of $\mathcal{J}(p)$. Furthermore, all these preimages are distinct. Therefore, every value $x \in [\xi, \eta]$ has exactly $d$ distinct preimages in $[a, b]$ by continuity of $p$. Since $\varepsilon$ was arbitrary and the two points $a, b$ also have all their preimages in $\mathcal{J}(p) \subset [a, b]$, we have proved (4.1).

Assume on the other hand that $[a, b]$ satisfies (4.1). Since the map $p$ has only finitely many critical values, there exists $x \in [a, b]$ such that the backward iterates of $x$ are dense in the Julia set. By (4.1) all these backward iterates are real; therefore $\mathcal{J}(p)$ is real. $\square$

*Remark* 4·1. By the above proof we can always assume $[a, b] = [\min \mathcal{J}(p), \max \mathcal{J}(p)]$. Furthermore, we have

$$p(\{\min \mathcal{J}(p), \max \mathcal{J}(p)\}) \subseteq \{\min \mathcal{J}(p), \max \mathcal{J}(p)\},$$

which implies that at least one of the two end points of this interval is either a fixed point, or they form a cycle of length 2.

THEOREM 4·1. *Let $p$ be a polynomial of degree $d > 1$ with real Julia set $\mathcal{J}(p)$. Then for any fixed point $\xi$ of $p$ with $\min \mathcal{J}(p) < \xi < \max \mathcal{J}(p)$ we have $|p'(\xi)| \geq d$. Furthermore, $|p'(\min \mathcal{J}(p))| \geq d^2$ and $|p'(\max \mathcal{J}(p))| \geq d^2$. Equality in one of these inequalities implies that $p$ is linearly conjugate to the Chebyshev polynomial $T_d$ of degree $d$.*

*Remark* 4·2. This theorem can be compared to [**8**, Theorem 2] and [**29**, **38**], where estimates for the derivative of $p$ for connected Julia sets are derived. Furthermore, in [**13**] estimates for $\frac{1}{n} \log |(p^{(n)})'(z)|$ for periodic points of period $n$ are given.

Before we give a proof of the theorem, we present a lemma, which is of some interest on its own. A similar result is given in [**27**, Chapter V, Section 2, Lemma 3].

LEMMA 4·2. *Let $f$ be holomorphic in the angular region $W_{\alpha,\beta}$ If there exists a positive constant $M$ such that*

$$\forall z \in W_{\alpha,\beta} : |f(z)| \geq M,$$

*then*

$$\forall \varepsilon > 0 \ \exists A, B > 0 \ \forall z \in W_{\alpha+\varepsilon,\beta-\varepsilon} : |f(z)| \leq B \exp(A|z|^{\kappa})$$

*with $\kappa = \frac{\pi}{\beta-\alpha}$.*

*Proof.* Without loss of generality we can assume that $M = 1$, $\alpha = -\frac{\pi}{2}$, and $\beta = \frac{\pi}{2}$. In this case $\kappa = 1$. The function

$$v(z) = \log |f(z)|$$

is a positive harmonic function in the right half-plane. Thus it can be represented by the Nevanlinna formula (cf. [**28**, p.100])

$$v(x+iy) = \frac{x}{\pi} \int_{-\infty}^{\infty} \frac{d\nu(t)}{|z-it|^2} + \sigma x, \tag{4.2}$$

where $\nu$ denotes a measure satisfying

$$\int_{-\infty}^{\infty} \frac{d\nu(t)}{1+t^2} < \infty$$

and $\sigma \geq 0$.

In the region given by $|\arg z| \leq \frac{\pi}{2} - \varepsilon$ and $|z| > 1$ we have

$$|z - it| \geq \max(|t| \sin\varepsilon, |z| \sin\varepsilon) \geq \max(1, |t|) \sin\varepsilon.$$

From this it follows that

$$|z - it|^2 \geq \frac{1}{2}(1+t^2) \sin^2 \varepsilon,$$

which gives

$$\int_{-\infty}^{\infty} \frac{d\nu(t)}{|z-it|^2} \leq \frac{2}{\sin^2\varepsilon} \int_{-\infty}^{\infty} \frac{d\nu(t)}{1+t^2} \leq B_\varepsilon$$

for $|z| \geq 1$ and some $B_\varepsilon > 0$. Setting $A = \frac{1}{\pi} B_\varepsilon + \sigma$ and observing that $x \leq |z|$ completes the proof. $\square$

*Proof of Theorem 4·1* Without loss of generality we may assume that the fixed point $\xi = 0$. Then we consider the solution $f$ of the Poincaré equation

$$f(\lambda z) = p(f(z))$$

with $\lambda = p'(0)$. We assume first that $\lambda > 0$.

First we consider the case $\min \mathcal{J}(p) < \xi < \max \mathcal{J}(p)$. In this case the function $f(z)/z$ tends to infinity uniformly for $z \to \infty$ in the region $\varepsilon \leq \arg z \leq \pi - \varepsilon$ for any $\varepsilon > 0$ by Theorem 2·1. Furthermore, we know that

$$|f(z)| \geq C \exp(A|z|^{\log_\lambda d})$$

in this region for some positive constants $A$ and $C$. Since $f(z)/z$ does not vanish at $z = 0$, this function satisfies the hypothesis of Lemma 4·2, from which we derive that

$$\log_\lambda d \leq \frac{\pi}{\pi - 2\varepsilon}$$

holds for any $\varepsilon > 0$, which implies $\lambda = p'(0) \geq d$.

The proof in the case $\xi = \max \mathcal{J}(p)$ runs along the same lines. The function $f(z)/z$ tends to infinity uniformly in any region $|\arg z| \leq \pi - \varepsilon$ in this case, which by Lemma 4·2 implies

$$\log_\lambda d \leq \frac{\pi}{2\pi - 2\varepsilon}$$

for all $\varepsilon > 0$, and consequently $\lambda = p'(0) \geq d^2$.

For negative $\lambda = p'(0)$ we apply the same arguments to $p^{(2)}$.

For the proof of the second assertion of the theorem, we first assume that the fixed point $\xi = 0$ satisfies $a = \min \mathcal{J}(p) < 0 < \max \mathcal{J}(p) = b$ and that $p'(0) = d$. We know that for a suitable linear conjugate $q$ of the Chebyshev polynomial $T_d$ we have $q'(0) = d$ and $\mathcal{J}(q) = [a, b]$ with $0 \in (a, b)$.

Let us assume now that $p'(0) = d$ and $\mathcal{J}(p)$ is a Cantor subset of the real line, or after a rotation that $\mathcal{J}(p)$ is a Cantor subset of the imaginary axis (this makes notation slightly simpler).

By arguments, similar to those in the beginning of Section 2·3 we can write

$$H(z) = \Re \log g(f(z)) = \int_{\mathcal{J}(p)} \log |f(z) - x| \, d\mu(x). \tag{4.3}$$

Since $\Re \log g(.)$ is the Green function of $\mathcal{J}(p)$ with pole at $\infty$ (cf. [**3**, Lemma 9.5.5] or [**39**]), we know that $H(z) \geq 0$ for all $z \in \mathbb{C}$ and $H(z) = 0$, if and only if $f(z) \in \mathcal{J}(p)$ (since $\mathcal{K}(p) = \mathcal{J}(p)$ in the present case). By Theorem 2·1 we have

$$H(z) = \Re\left(zF(\log_d z)\right) = x\Re(F(\log_d z)) - y\Im(F(\log_d z)) \text{ for } z = x + iy, \tag{4.4}$$

and by Theorem 2·2 the function $F$ is not constant in the present case. The periodic function $\Im F(t + i\varphi)$ has zero mean, since the mean of $F$ is real. Thus $\Im F(t + i\varphi)$ attains positive and negative values for any $\varphi$. We now take $z = iy \in i\mathbb{R}^+$ to obtain

$$H(iy) = -y\Im(F(\log_d y + i\frac{\pi}{2\log d})).$$

Since $\Im F$ attains positive values by the above argument, we get a contradiction to $H(z) \geq 0$ for all $z$.

A similar argument shows that for $0 = \max \mathcal{J}(p)$ and $p'(0) = d^2$ the assumption that the Julia set is not an interval leads to the same contradiction. □

*Remark* 4·3. Lemma 6.4 in [**9**] proves Theorem 4·1 for the special case of quadratic polynomials. The proof given in [**9**] is purely geometrical.

*Remark* 4·4. We have a purely real analytic proof for $|p'(\max \mathcal{J}(p))| \geq d^2$, which is motivated by the proof of the extremality of the Chebyshev polynomials of the first kind given in [**40**]. However, we could not find a similar proof for the other assertions of the theorem.

### 4·1. *The Julia set is a subset of the negative reals*

As a consequence of Lemma 4·2 we get that any solution of the Poincaré equation for a polynomial with Julia set contained in the negative real axis has order $\leq \frac{1}{2}$. The only solutions of a Poincaré equation with order $\frac{1}{2}$ in this situation are the functions

$$f(z) = \frac{1}{a}\left(\cosh\sqrt{2az} - 1\right)$$

for

$$p(z) = (T_d(az + 1) - 1)/a,$$

where $a \in \mathbb{R}^+$ and $T_d$ denotes the Chebyshev polynomial of the first kind of degree $d$. This is also the only case where the periodic function $F$ in (2.6) is constant in this situation.

COROLLARY 4·1. *Assume that $p$ is a real polynomial such that $\mathcal{J}(p)$ is real and all coefficients $p_i$ $(i \geq 2)$ of $p$ are non-negative. Then $\mathcal{J}(p) \subset \mathbb{R}^- \cup \{0\}$ and therefore*

$$f(z) \sim \exp\left(z^\rho F\left(\frac{\log z}{\log \lambda}\right)\right) \tag{4.5}$$

*for $z \to \infty$ and $|\arg z| < \pi$. Here $F$ is a periodic function of period 1 holomorphic in the strip given by $|\Im w| < \frac{\pi}{\log \lambda}$. Furthermore, for every $\varepsilon > 0$ $\Re e^{i\rho \arg z} F(\frac{\log z}{\log \lambda})$ is bounded between two positive constants for $|\arg z| \leq \pi - \varepsilon$.*

*Proof.* From [**9**, Lemmas 6.4 and 6.5] it follows that $f(z)$ has only non-positive real zeros. Then by Theorem 3·2 $\mathcal{J}(p) \subset \mathbb{R}^- \cup \{0\}$. Finally, the assertion follows by applying [**9**, Theorem 7.5]. $\square$

*Example* 1. In order to illustrate the above results, we shall turn to the equation

$$f(5z) = 4f(z)^2 - 3f(z),$$

which arises in the description of Brownian motion on the Sierpiński gasket [**10**, **24**, **25**, **43**]. Here $p(z) = 4z^2 - 3z$, and the fixed point of interest is $f(0) = 1$. This fits into the assumptions of Section 1·2 only after substituting $g(z) = 4(f(z) - 1)$, where $g$ satisfies

$$g(5z) = g(z)^2 + 5g(z).$$

Now Corollary 4·1 may be applied to this equation (the preimages of 0 are real by [**9**, Lemma 6.7]) to give (4.5).

Note also that $p'(0) = 5 > 4 = 2^2$ in accordance with Theorem 4·1.

4·2. *The Julia set has positive and negative elements*

Again as a consequence of Theorem 4·1 the solution of the Poincaré equation for a polynomial with real Julia set with positive and negative elements has order $\leq 1$. The only solution of a Poincaré equation of order 1 in this situation are the functions

$$f(z) = \frac{1}{a}\left(\cos\left(a\frac{z - \frac{2k\pi}{d-1}}{\sin\frac{k\pi}{d-1}}\right) - \xi_k\right)$$

for

$$p(z) = \frac{1}{a}\left(T_d(a(z + \xi_k)) - \xi_k\right),$$

where $a \in \mathbb{R}^+$ and $\xi_k = \cos\frac{k\pi}{d-1}$ for $1 \leq k < \frac{d-1}{2}$. This is again the only case where the periodic function $F$ in (2.6) is constant in this situation.

## 5. *The Zeta function of the Poincaré function*

In [**10**] the zeta function of a fractal Laplace operator was related to the zeta function of certain Poincaré functions. Asymptotic expansions for the Poincaré functions were then used to give a meromorphic continuation of these zeta functions as well as information on the location of their poles and values of residues. In this section we give a generalisation of these results to polynomials whose Fatou set contains an angular region $W_{-\alpha,\alpha}$ around the positive real axis. In this case the solution $f$ of (1.2) has no zeros in an angular region $W_{-\alpha,\alpha}$. Furthermore, from the Hadamard factorisation theorem we get

$$f(z) = z\exp\left(\sum_{\ell=1}^{k}(-1)^{\ell-1}\frac{e_\ell z^\ell}{\ell}\right)\prod_{\substack{f(-\xi)=0\\ \xi\neq 0}}\left(1+\frac{z}{\xi}\right)\exp\left(-\frac{z}{\xi}+\frac{z^2}{2\xi^2}+\cdots+(-1)^{k-1}\frac{z^k}{k\xi^k}\right), \tag{5.1}$$

where $k = \lfloor \log_\lambda d \rfloor$. By the discussion in [**10**, Section 5] the values $e_1, \ldots, e_k$ are given by the first $k$ terms of the Taylor series of $\log\frac{f(z)}{z}$

$$\log\frac{f(z)}{z} = \sum_{\ell=1}^{k}(-1)^{\ell-1}\frac{e_\ell z^\ell}{\ell} + \mathcal{O}(z^{k+1}).$$

The zeta function of $f$ is now defined as

$$\zeta_f(s) = \sum_{\substack{f(-\xi)=0\\ \xi\neq 0}} \xi^{-s}, \tag{5.2}$$

where $\xi^{-s}$ is defined using the principal value of the logarithm, which is sensible, since $\xi$ is never negative real by our assumption on $\mathcal{F}_\infty(p)$. The function $\zeta_f(s)$ is holomorphic in the half plane $\Re s > \rho$. In [**10**] we used the equation

$$\int_0^\infty \left(\log f(x) - \log x - \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{e_\ell x^\ell}{\ell}\right) x^{-s-1}\, dx = \zeta_f(s) \frac{\pi}{s \sin \pi s}, \tag{5.3}$$

which holds for $\rho < \Re s < k+1$, to derive the existence of a meromorphic continuation of $\zeta_f$ to the whole complex plane. There ([**10**, Theorem 8]) we obtained

$$\operatorname*{Res}_{s=\rho+\frac{2k\pi i}{\log\lambda}} \zeta_f(s) = -\frac{f_k}{\pi}\left(\rho + \frac{2\pi i k}{\log \lambda}\right) \sin \pi \left(\rho + \frac{2\pi i k}{\log\lambda}\right),$$

where $f_k$ is given by (2.27). From this we get

$$\operatorname*{Res}_{s=\rho+\frac{2k\pi i}{\log\lambda}} \zeta_f(s) = -\operatorname*{Res}_{s=-\rho-\frac{2k\pi i}{\log\lambda}} M_\mu(s). \tag{5.4}$$

This shows that the function

$$\zeta_f(s) - M_\mu(-s) \tag{5.5}$$

is holomorphic in $\rho - 1 < \Re s < \rho + 1$, since the single poles on the line $\Re s = \rho$ cancel. This fact was used in [**18**] to derive an analytic continuation for $\zeta_f(s)$.

THEOREM 5·1. *Let $f$ be the entire solution of* (1.2) *and assume that $p$ is neither linearly conjugate to a Chebyshev polynomial nor to a monomial and that $W_{-\alpha,\alpha} \subset \mathcal{F}_\infty(p)$ for some $\alpha > 0$. Then the following assertions hold*

(i) *the limit $\lim_{t\to\infty} t^{-\rho} \log f(t)$ does not exist.*
(ii) *$\zeta_f(s)$ has at least two non-real poles in the set $\rho + 2\pi i \sigma \mathbb{Z}$ ($\sigma = \frac{1}{\log\lambda}$).*
(iii) *the limit $\lim_{x\to 0} x^{-\rho} G(x)$ with $G$ given by* (2.18) *does not exist.*

*Proof.* Equation (2.6) in Theorem 2·1 (see also [**9**]) implies that

$$z^{-\rho} \log f(z) = F(\log_\lambda z) + o(1) \text{ for } z \to \infty \text{ and } z \in W_{-\alpha,\alpha}$$

with a periodic function $F$ of period 1. Theorem 2·2 implies that $F$ is a non-constant . Thus the limit in (i) does not exist.

Since the periodic function $F$ is non-constant, there exists a $k_0 \neq 0$ such that the Fourier-coefficients $f_{\pm k_0}$ do not vanish. By (2.6) we have

$$\log f(z) = z^\rho \sum_{k\in\mathbb{Z}} f_k z^{\frac{2k\pi i}{\log \lambda}} + \mathcal{O}(z^{-M})$$

for any $M > 0$. By properties of the Mellin transform (cf. [**35**]), every term $Az^{\rho+i\tau}$ in the asymptotic expansion of $\log f(z)$ corresponds to a first order pole of the Mellin transform of $\log f(z)$ with residue $A$ at $s = \rho + i\tau$. Since $f_{k_0} \neq 0$, from (5.3) we have simple poles of $\zeta_f(s)$ at $s = \rho \pm \frac{2k_0\pi i}{\log\lambda}$.

Assertion (iii) follows from (i) by (2.19). $\square$

In the following we consider the zero counting function of $f$

$$N_f(x) = \sum_{\substack{|\xi|<x\\ f(\xi)=0}} 1. \tag{5.6}$$

Theorem 5·2. *Let $f$ be the entire solution of* (1.2). *Then the following are equivalent*
(i) *the limit* $\lim_{x\to\infty} x^{-\rho} N_f(x)$ *does not exist.*
(ii) *the limit* $\lim_{t\to 0} t^{-\rho}\mu(B(0,t))$ *does not exist.*

*Proof.* For the proof of the equivalence of (i) and (ii) we observe that by the fact that $f'(0)=1$, there is an $r_0>0$ such that $f: B(0,r_0)\to\mathbb{C}$ is invertible. For the following we choose $n = \lfloor \log_\lambda(x/r_0)\rfloor + k$ and let the integer $k>0$ be fixed for the moment. Then we use the functional equation for $f$ to get

$$N_f(x) = \#\left\{\xi \mid f(\lambda^n\xi) = p^{(n)}(f(\xi)) = 0 \wedge |\xi| < x\lambda^{-n}\right\} = \#\left(p^{(-n)}(0)\cap f(B(0,x\lambda^{-n}))\right).$$

This last expression can now be written in terms of the discrete measure $\mu_n$ given in (2.15)

$$N_f(x) = d^n\mu_n\left(f(B(0,x\lambda^{-n}))\right).$$

By the weak convergence of the measures $\mu_n$ (cf. [**7**]) we get for $x\to\infty$ (equivalently $n\to\infty$)

$$N_f(x) = d^n\mu(f(B(0,x\lambda^{-n}))) + o(d^n) = x^\rho(x\lambda^{-n})^{-\rho}\mu(f(B(0,x\lambda^{-n}))) + o(x^\rho).$$

By our choice of $n$ we have $r_0\lambda^{-k-1}\le x\lambda^{-n}\le r_0\lambda^{-k-1}$, which makes the first term dominant. From this it is clear that the existence of the limit

$$\lim_{x\to\infty} x^{-\rho}N_f(x) = C$$

is equivalent to

$$\mu(f(B(0,t))) = Ct^\rho \text{ for } r_0\lambda^{-k}\le t < r_0\lambda^{-(k-1)}.$$

Since $k$ was arbitrary this implies

$$\mu(f(B(0,t))) = Ct^\rho \text{ for } 0<t<r_0. \tag{5.7}$$

It follows from $f'(0)=1$ that

$$\forall\varepsilon>0:\exists\delta>0:\forall t<\delta: B(0,(1-\varepsilon)t)\subset f(B(0,t))\subset B(0,(1+\varepsilon)t). \tag{5.8}$$

Thus the existence of the limit in assertion (ii) is equivalent to

$$\lim_{t\to 0} t^{-\rho}\mu(f(B(0,t))) = C.$$

Thus (i) and (ii) are equivalent. □

*Remark* 5·1. If $\mathcal{J}(p)$ is real and disconnected then the limits in Theorem 5·2 do not exist. Furthermore, it is known that the limit

$$\lim_{t\to 0} t^{-\rho}\mu(f(B(w,t))) = C$$

does not exist for $\mu$-almost all $w\in\mathcal{J}(p)$ (cf. [**32**, Theorem 14.10]), if $\rho$ is not an integer.

This motivates the following conjecture.

Conjecture. *The limits in Theorem 5·2 exist, if and only if $p$ is either linearly conjugate to a Chebyshev polynomial or a monomial.*

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